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The following work has been completed: The recovery of irregularly samples band-limited functions via tempered splines. Band limited functions can be recovered from their values on certain irregularly distributed discrete sampling sets as the limits of the piecewise polynomial spline interpolants when the order of the splines goes to infinity. This significant extension of the classical case when the sampling set is a lattice which was considered by L. Collatz, W. Quade, I.J. Schoenberg, and others. Orthogonality criteria for compactly supported scaling functions. The question of whether the integer translates of the scaling function constructed from a prescribed scaling sequency in the standard way are mutually orthogonal is quite subtle. The various conditions and the supporting arguments which are currently in the literature are very complicated.

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Multivariate Wavelet Representations and Approximations

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Report Summary

The principal investigator, W. R. Madych, participated in the AMS-SIAM Summer Seminar on the mathematics of tomography, impedance imaging, and integral geometry held at Mt. Holyoke, June, 1993. He gave a lecture on his recent work in wavelets and sampling at this seminar.

In the preceding reports we outlined several lines of investigation which we are pursuing. The following work has been completed since the last report:

- *The recovery of irregularly sampled band-limited functions via tempered splines.* In this article we show that band limited functions can be recovered from their values on certain irregularly distributed discrete sampling sets as the limits of the piecewise polynomial spline interpolants when the order of the splines goes to infinity. This is significant extension of the classical case when the sampling set is a lattice which was considered by L. Collatz, W. Quade, I. J. Schoenberg, and others.

- *Orthogonality criteria for compactly supported scaling functions.* The question of whether the integer translates of the scaling function constructed from a prescribed scaling sequence in the standard way are mutually orthogonal is quite subtle. The various conditions and the supporting arguments which are currently in the literature are very complicated. In this article we give new simple proofs of several criteria for the orthogonality of the integer translates of a scaling function.

Both articles are included in this report.

At present we are preparing several papers detailing our grant related work. The topics include (i) certain questions concerning irregular sampling of signals, (ii) results concerning the breakdown the so-called scaling functions into more elementary building blocks, and (iii) results concerning multivariate wavelets and tilings of R^n.

Finally we mention that in addition to the ongoing work we we have completed twenty one articles on various aspects of wavelets and their applications. All of these have been included together with detailed summaries in the five semi-annual and annual technical reports which have been submitted. As of this date fifteen have been published or accepted for publication in various scholarly journals and books.

THE RECOVERY OF IRREGULARLY SAMPLED BAND-LIMITED FUNCTIONS VIA TEMPERED SPLINES

Yu. Lyubarskii and W. R. Madych

Abstract

We show that band limited functions can be recovered from their values on certain irregularly distributed discrete sampling sets as the limits of the piecewise polynomial spline interpolants when the order of the splines goes to infinity. This is an extension of the classical case when the sampling set is a lattice which was considered by L. Collatz, W. Quade, I. J. Schoenberg, and others.

1 Introduction

1.1 Overview

This paper concerns univariate splines of even order and band-limited functions. The main result asserts that functions in the classical Paley-Wiener class PW_π, that is, those square integrable functions whose Fourier transforms are supported in the interval $[-\pi,\pi]$, can be recovered from their values on certain irregularly spaced sampling sequences via the formula

$$\lim_{k\to\infty} s_k(x) = f(x)$$

where s_k is the spline of order $2k$ which interpolates f on such a sequence. For more precise statements see Theorems 4 and 5 in Section 4.3 below. The sampling sequences $\{x_n\}$ considered here are those for which the corresponding exponential functions $\{e^{-ix_n\xi}\}$ constitute a Riesz basis for $L^2([-\pi,\pi])$.

1.2 Background, motivation, and objectives

If f is a function in the Paley-Wiener class then it is uniquely determined by its values on the integer lattice $\mathbb{Z}$ and can be recovered from these values via the Whittaker-Kotelnikov-Shannon sampling formula

$$f(x) = \sum_{n=-\infty}^{\infty} f(n)\frac{\sin \pi(x-n)}{\pi(x-n)}. \tag{1}$$

For example see [27].

Such f's are also uniquely determined and can be recovered from their values on certain irregular sampling sets $\{x_n\} = \{\ldots x_{-1}, < x_0 < x_1 < \ldots\}$. In particular if $\{x_n\}$ is such that the collection of functions $\varphi_n(\xi) = e^{ix_n\xi}$, $n \in \mathbb{Z}$, is a Riesz basis for $L^2([-\pi, \pi])$ then the natural substitute for (1) is

$$f(x) = \sum_{n=-\infty}^{\infty} f(x_n)\frac{S(x)}{S'(x_n)(x-x_n)} \tag{2}$$

where $S(z)$ is the unique entire function of exponential type π whose zero set is $\{x_n\}$, see [19]. For more references see the discussion in Section 5.

Because $\sin \pi x$ as well as $S(x)$ fail to belong to $L^2(-\infty, \infty)$ the cardinal functions

$$\frac{\sin \pi(x-n)}{\pi(x-n)} \quad \text{and} \quad \frac{S(x)}{S'(x_n)(x-x_n)}$$

have relatively slow decay as $x \to \pm\infty$. This makes formulas (1) and (2) somewhat unstable. A perturbation at n or x_n will have significant influence at points x far from n or x_n respectively.

Schoenberg used cardinal splines to stabilize (1) in [23]. Subsequently, he showed that if f is in the Paley-Wiener class PW_r and s_k is the piecewise polynomial spline of order $2k$ which interpolates f on $\mathbb{Z}$ then

$$\lim_{k\to\infty} s_k(x) = f(x). \tag{3}$$

uniformly in x. The spline s_k enjoys the representation

$$s_k(x) = \sum_{n=-\infty}^{\infty} f(n)\lambda_k(x-n), \tag{4}$$

where $\lambda_k(x)$ is the corresponding cardinal spline function. This fundamental spline λ_k is the unique spline of polynomial growth of order $2k$ with knots on $\mathbb{Z}$ which satisfies $\lambda_k(n) = \delta_{0,n}$, $n \in \mathbb{Z}$, where $\delta_{0,n}$ is the Kronecker delta. Since λ_k has exponential decay at $\pm\infty$, see [23], (3) and (4) represent a more stable method of approximating f than (1). For extension to wider classes of band-limited f's see [24, 22, 18]. The results make essential use of the lattice structure of the sampling set $\mathbb{Z}$, particularly the Poisson summation formula.

It should be mentioned that the use of splines in a summability method for the recovery of regularly sampled band limited functions goes at least as far back as the work of Quade and Collatz [21] where a variant of (3) was established for certain trigonometric polynomials f. See also [24, page 103].

In the case of irregularly spaced knot or sampling sequences $\{x_n\}$ that satisfy certain natural conditions the corresponding cardinal functions which build the interpolating piecewise polynomial splines s_k are also known to enjoy exponential decay, see [3]. Convenient ways of evaluating such splines $s_k(t)$ in terms of the data $\{f(x_n)\}$ can be found, for example, in [4].

In view of this situation it is natural to ask whether the spline summability method, which makes essential use of the lattice structure of the sampling set $\mathbb{Z}$, can be extended to the case of irregularly sampled data. As indicated in the overview, Section 1.1, we found the answer to this query to be yes. To obtain this answer we applied the technique of mean periodic continuation with respect to a given basis of exponentials to the study of splines with a biinfinite knot sequence. The goal of this article is to outline the theory and to show how this machinery works.

To maintain clarity and avoid the use of obfuscating and unessential technical details we have restricted our reconstruction results to the classical Paley-Wiener class.

1.3 Contents and notation

This paper is organized as follows:

Section 2 is devoted to a brief summary of certain aspects of the theory of tempered splines which are germane to later developments in this paper. Properties and examples of Riesz bases for $L^2([-\pi,\pi])$ consisting of exponentials $\{e^{-ix_n\xi}\}$ as well as notions and results concerning mean periodic continuation are reviewed in Section 3. The main results together with detailed proofs are contained in Section 4. Various remarks pertaining to the material found here, citations, and acknowledgements are collected together in Section 5.

We use standard mathematical notation and only alert the reader that all Fourier transforms are interpreted in the distributional sense with the normalization that when f is an integrable function then its Fourier transform $\hat{f}$ is given by

$$\hat{f}(\xi) = \int_{-\infty}^{\infty} f(x)e^{-ix\xi}dx.$$

As is customary, the symbols c and C denote generic constants whose values depend on their occurrence and are usually independent of the essential parameters in question.

2 Tempered Splines

2.1 Knot sequences

The type of knot or sampling sequences we consider in this section are sequences of real numbers $\{x_n\}_{n\in\mathbb{Z}} = \{\ldots x_{-1} < x_0 < x_1 < x_2 < \ldots\}$ which satisfy the following properties:

(*i*) A sequence $\{x_n\}$ is *sufficiently dense* if there is a positive number r, such that intervals of length $2r$ centered at x_n, $n \in \mathbb{Z}$, cover all of the real line $\mathbb{R}$. In other words

$$\mathbb{R} = \bigcup_{n\in\mathbb{Z}} \left\{x : \; |x - x_n| \leq r\right\}.$$

(*ii*) A sequence $\{x_n\}$ is *separated* if there is a positive number δ such that $|x_n - x_m| \geq \delta$ whenever $n \neq m$.

A sequence $\{x_n\}_{n\in\mathbb{Z}}$ is said to satisfy condition SDS whenever it satisfies both properties (*i*) and (*ii*). Throughout this section we will always assume that $\{x_n\}_{n\in\mathbb{Z}}$ is a sequence which satisfies condition SDS.

2.2 Splines of even order

Splines were studied by Schoenberg [23] in the case of the knot sequence $\{x_n\} = \mathbb{Z}$ and subsequently investigated and further developed by many authors; see Section 5 for more references. Here we find it convenient to restrict our attention to the class of those splines which are also tempered distributions. This is a natural way of eliminating certain pathological examples and allows us to use various distributional machinery including the Fourier transform.

Suppose $\{x_n\}_{n\in\mathbb{Z}}$ is a sequence of real numbers which satisfies condition SDS. A *tempered spline of order* $2k$ *with knot sequence* $\{x_n\}_{n\in\mathbb{Z}}$ is a tempered distribution s which satisfies

$$D^{2k}s(x) = \sum_{n\in\mathbb{Z}} a_n \delta(x - x_n) \tag{5}$$

where D is the differential operator $\dfrac{d}{dx}$, $D^{2k} = \dfrac{d^{2k}}{dx^{2k}} = DD^{2k-1}$, $\{a_n\}_{n\in\mathbb{Z}}$ is a sequence of constants, and $\delta(x)$ is the Dirac measure at the origin. The class of all tempered splines of order $2k$ with knot sequence $\{x_n\}$ is denoted by $SH_k(\{x_n\})$.

The following proposition follows from arguments identical to that used to prove the corresponding fact in the case $\{x_n\} = \mathbb{Z}$. Details may be found in Section 2 of [17].

Proposition 1 *The following statements are equivalent*

- *s is in $SH_k(\{x_n\})$*
- *s is a function which satisfies the following properties:*

  (*i*) *s is in $C^{2k-2}(R)$.*

  (*ii*) *s is of polynomial growth. That is, there are positive constants c and p such that*
  $$|s(x)| \leq c(1 + |x|)^p.$$

  (*iii*) *On the complement of $\{x_n\}$, $R \backslash \{x_n\}$, the function s is infinitely differentiable and satisfies $D^{2k}s = 0$.*

Thus $SH_k(\{x_n\}_{n\in\mathbb{Z}})$ is simply the familiar class of piecewise polynomial splines of order $2k$ which are of polynomial growth and have the knot sequence $\{x_n\}$.

2.3 The interpolation problem and its solution

The natural problem of interpolation for tempered splines is the following:

Given a sequence of real or complex numbers $\{y_n\}_{n\in\mathbb{Z}}$ find an element s in $SH_k(\{x_n\})$ such that $s(x_n) = y_n$ for all n in $\mathbb{Z}$.

Since elements of $SH_k(\{x_n\})$ are of polynomial growth it is clear that in order for this problem to have a solution a necessary requirement on the sequence $\{y_n\}$ is that it also be of polynomial growth. It turns out that this condition is also sufficient.

The following result is a basic ingredient in the solution of this problem.

Theorem 1 (de Boor [3]) *Suppose k is a positive integer and $\{x_n\}$ is a sequence which satisfies condition SDS. Then for each integer m, $m \in \mathbb{Z}$, there is an element λ_m in $SH_k(\{x_n\})$ which satisfies the following properties:*

1. *For each element x_n in $\{x_n\}$ the function λ_m satisfies*
$$\lambda_m(x_n) = \delta_{m,n}$$
where $\delta_{m,n}$ is the Kronecker delta.

2. *There exist positive constants C and c which depend only on k and the knot sequence $\{x_n\}$, such that*

$$|\lambda_m(x)| \leq Ce^{-c|x-x_m|},$$

for all $m \in \mathbb{Z}$.

Because of the first property listed above the functions λ_m are the fundamental functions of interpolation. In analogy with classical polynomial interpolation or with the *sinus cardinalis* basis, $\dfrac{\sin \pi(x-m)}{\pi(x-m)}$, the λ_m's are often referred to as the Lagrange functions or the cardinal functions.

Consider any sequence $\{y_n\}$ which is of polynomial growth and the cardinal splines λ_m, $m \in Z$, in $SH_k(\{x_n\})$ whose existence is guaranteed by the above theorem. The function

$$s(x) = \sum_{n\in\mathbb{Z}} y_n \lambda_n(x) \tag{6}$$

is well defined for all real x since the series converges uniformly on compact subsets of the real line $\mathbb{R}$. Note that s is a member of $SH_k(\{x_n\})$ which satisfies $s(x_n) = y_n$, for all n in $\mathbb{Z}$. Thus s is a solution to the interpolation problem for the data $\{y_n\}$. This solution is unique since each non-zero spline of order $2k$ with knot sequence $\{x_n\})$ satisfying $s(x_n) = 0$ for all $n \in \mathbb{Z}$ must have exponential growth in at least one direction; see the remark preceding Theorem 3 in [3].

We summarize these observations as follows:

Proposition 2 *Suppose k is a positive integer, $\{x_n\}$ is a sequence which satisfies condition SDS, and $\{y_n\}$ is a sequence which for some real number p satisfies*

$$y_n = 0(|x_n|^p) \quad as \quad n \to \pm\infty.$$

Then there is a unique element s in $SH_k(\{x_n\})$ which satisfies

$$s(x_n) = y_n$$

for all n in $\mathbb{Z}$. Furthermore this element s enjoys the representation (6) and satisfies

$$s(x) = 0(|x|^p) \quad as \quad x \to \pm\infty.$$

3 Riesz bases and mean periodic continuation

3.1 Riesz bases consisting of exponential functions

Recall that a basis $\{\varphi_n\}$ of a Hilbert space $\mathcal{H}$ is called a Riesz basis if for every linear combination $f = \sum_n a_n \varphi_n$ in $\mathcal{H}$

$$c \sum_n |a_n|^2 \leq \|f\|^2 \leq C \sum_n |a_n|^2 \tag{7}$$

where c and C are positive constants independent of f.

Given a sequence $\{z_n\}_{n\in\mathbb{Z}}$ of complex numbers the collection of functions φ_n defined by

$$\varphi_n(\xi) = e^{-iz_n\xi}, \tag{8}$$

$-\infty < \xi < \infty$, $n \in \mathbb{Z}$, is denoted by $\mathcal{E}(\{z_n\})$. Thus

$$\mathcal{E}(\{z_n\}) = \{\varphi_n\}_{n\in\mathbb{Z}} \tag{9}$$

where the function φ_n are defined by (8).

Properties of the collections $\mathcal{E}(\{z_n\})$ as bases where first studied by Paley and Wiener [19] and subsequently investigated by many authors; see Section 5 for more references and further comments. Here our objective is to provide access to a sufficiently rich class of examples of sequences $\{z_n\}$ such that the corresponding $\mathcal{E}(\{z_n\})$'s are Riesz bases for $L^2([-\pi, \pi])$. Toward this end we list two results: one is formulated in terms of the geometric proximity of the points $\{z_n\}$ to the integers while the other is formulated in terms of zeros $\{z_n\}$ of entire functions from a special class introduced by Levin [15, 16] which, in analogy with the case $\{z_n\} = \mathbb{Z}$, are referred to as sine type functions.

Theorem 2 (Kadets [12]) *If $z_n = n + w_n$, $n \in \mathbb{Z}$, where $\sup_n |w_n| < \infty$ and $|Re\ w_n| \leq r < \frac{1}{4}$ then $\mathcal{E}(\{z_n\})$ is a Riesz basis for $L^2([-\pi, \pi])$.*

To state the second result we need to review the following definition.

An entire function $S(z)$ of exponential type is said to be of *sine type* if it satisfies the following properties:

1. All its zeros $\{z_n\}$ are simple and lie in a horizontal strip. In other words

$$\{z_n\} \subset \{z : |Im z| < y_0\}$$

for some positive number y_0.

2. The set of zeros $\{z_n\}$ is separated, that is, there is a positive number δ such that $|z_m - z_n| \geq \delta$ whenever $m \neq n$.

3. For some $y > y_0$ and positive constants m and M

$$m < |S(x+iy)| < M$$

for all x, $-\infty < x < \infty$.

4.

$$\limsup_{y \to \pm\infty} \frac{\log |S(iy)|}{|y|} = \pi.$$

Theorem 3 (Levin [15], Golovin [8]) *If $\{z_n\}$ is the set of zeros of a sine type function then the collection $\mathcal{E}(\{z_n\})$ is a Riesz basis for $L^2([-\pi,\pi])$.*

Examples of sine type functions may be obtained by looking for representations of the form

$$S(z) = \int_{-\pi}^{\pi} e^{iz\xi} d\sigma(\xi)$$

where $\sigma(\xi)$ is a function of bounded variation on $[-\pi,\pi]$ having jumps at the endpoints $\pm\pi$. If the zeros of such a function are separated then $S(z)$ is a sine type function. A specific concrete example is given by

$$S(z) = \prod_{j=0}^{N-1} \sin \pi\left(\frac{z - a_j}{N}\right)$$

where the a_j's are positive numbers satisfying $0 \leq a_0 < a_1 < \ldots < a_{N-1} < N$; its zero set is $\cup_{j=0}^{N-1}(N\mathbb{Z} + \alpha_j) = \cup_{k \in \mathbf{Z}}\{\alpha_0 + Nk, \ldots, \alpha_{N-1} + Nk\}$, choosing small a_j's shows that the gaps in such a sequence can be very large.

Before leaving this section it should be mentioned that necessary and sufficient conditions for the collection $\mathcal{E}(\{z_n\})$ to be a Riesz basis for $L^2([-\pi,\pi])$ were obtained by Pavlov [20], see also [10]. These conditions are in terms of zeros of certain entire functions.

3.2 Sampling sequences

The kind of knot or sampling sequences we will be interested in for the rest of this paper are sequences of real numbers $\{x_n\}_{n \in \mathbf{Z}} = \{\ldots x_{-1} < x_0 < x_1 < x_2 < \ldots\}$ such that

(10) $\qquad \mathcal{E}(\{x_n\}_{n \in \mathbf{Z}})$ is a Riesz basis for $L^2([-\pi,\pi])$.

We call such sequences *RRB sequences* or say that it satisfies *condition* RRB. Thus, $\{x_n\}_{n\in\mathbf{Z}}$ is an RRB sequence if it is a sequence of real numbers and the corresponding sequence of exponentials $\mathcal{E}(\{x_n\}_{n\in\mathbf{Z}})$ enjoys property (10).

It is clear that every RRB sequences must be separated, that is, it must have property (ii) in Section 2.1. Otherwise it is easy to check that the left hand side inequality of (7) fails.

Furthermore it is a direct consequence one of Beurling's celebrated results, see [2, 14], that every RRB sequence $\{x_n\}_{n\in\mathbf{Z}}$ must satisfy

$$\liminf_{r\to\infty}\left\{\inf_{-\infty<a<\infty}\frac{\#(\{x_n\}_{n\in\mathbf{Z}}\cap(a,a+r))}{r}\right\}\geq 1 \tag{11}$$

where $(a,a+r)$ denotes the interval $a<x<a+r$ and $\#\Omega$ denotes the number of elements in the set Ω. It is clear that any sequence which satisfies (11) must be sufficiently dense, that is, it must satisfy property (i) in Section 2.1. Otherwise it is easy to check that the right hand side of inequality (11) is zero.

These observations are summarized as follows: Every RRB sequence satisfies condition SDS.

3.3 The operator of mean periodic continuation

Suppose that $\{x_n\}$ is an RRB sequence. Then each function h in $L^2([-\pi,\pi])$ enjoys the representation

$$h(\xi)=\sum_{n\in\mathbf{Z}}a_n\,e^{-ix_n\xi},\quad -\pi\leq\xi\leq\pi, \tag{12}$$

where the sequence of coefficients $\{a_n\}$ satisfies

$$c\|\{a_n\}\|_{\ell^2(\mathbf{Z})}\leq\|h\|_{L^2([-\pi,\pi])}\leq C\|\{a_n\}\|_{\ell^2(\mathbf{Z})} \tag{13}$$

where c and C are positive constants independent of h. Note that the series on the right hand side of (12) defines a prolongation of the function h to the whole real line. We call this prolongation H. Thus

$$H(\xi)=\sum_{n\in\mathbf{Z}}a_n\,e^{-ix_n\xi},\quad -\infty<\xi<\infty, \tag{14}$$

where the sequence of coefficients $\{a_n\}$ is the same as that in (12). By virtue of (13) the series (14) is locally L^2 convergent and the function $H(\xi)$ is locally in L^2.

Consider the linear operator $A: h \rightarrow Ah$ which maps $L^2([-\pi,\pi])$ into itself and is defined via

$$Ah(\xi) = \sum_{n\in\mathbb{Z}} a_n\, e^{-i2\pi x_n} e^{-ix_n\xi}, \quad -\pi \le \xi \le \pi, \tag{15}$$

where the coefficient sequence $\{a_n\}$ is uniquely determined by h via (12). Because the functions

$$\varphi_n(\xi) = e^{-ix_n\xi}, \; n \in \mathbb{Z} \tag{16}$$

are a Riesz basis for $L^2([-\pi,\pi])$ the operator A is well defined and invertible on $L^2([-\pi,\pi])$. Similarly the integers powers of A, $A^j = AA^{j-1}$, $j \in \mathbb{Z}$, may be defined via

$$A^j h(\xi) = \sum_{n\in\mathbb{Z}} a_n\, e^{-i2\pi j x_n}\, e^{-ix_n\xi}, \quad -\pi \le \xi \le \pi. \tag{17}$$

Again, because the collection of functions $\mathcal{E}(\{x_n\}) = \{\varphi_n\}$ defined by (16) is a Riesz basis the operator norms of A^j are uniformly bounded. namely,

$$\|A^j h\|_{L^2([-\pi,\pi])} \le C\|h\|_{L^2([-\pi,\pi])}, \; j \in \mathbb{Z} \tag{18}$$

where C is a constant independent of j and h.

Note that $A^j h$ can be expressed in terms of the prolongation H as follows:

$$A^j h(\xi) = H(\xi + 2\pi j), \quad -\pi \le \xi \le \pi.$$

For this reason A is referred to as the *prolongation operator* or the *operator of mean periodic continuation* with respect to the collection $\mathcal{E}(\{x_n\})$.

We will also need a representation of the adjoint A^* of A. To this end consider the Riesz basis $\{\varphi_n^*\}$ which is dual to $\{\varphi_n\}$ in $L^2([-\pi,\pi])$. Then if h in $L^2([-\pi,\pi])$ enjoys the representation

$$h(\xi) = \sum_{n\in\mathbb{Z}} a_n^* \varphi_n^*(\xi), \quad -\pi \le \xi \le \pi, \tag{19}$$

where the coefficient sequence $\{a_n^*\}$ is uniquely determined by h, it is clear that

$$A^* h(\xi) = \sum_{n\in\mathbb{Z}} a_n^*\, e^{i2\pi x_n}\, \varphi_n^*(\xi), \quad -\pi \le \xi \le \pi, \tag{20}$$

and, more generally, for all $j \in \mathbb{Z}$, $(A^*)^j = (A^j)^* = A^{*j}$ has the representation

$$A^{*j} h(\xi) = \sum_{n\in\mathbb{Z}} a_n^*\, e^{i2\pi j x_n}\, \varphi_n^*(\xi), \quad -\pi \le \xi \le \pi. \tag{21}$$

Since it will be used often in what follows, we remind the reader that the operator norms of $\{A^{*j}\}$ are also uniformly bounded.

4 The recovery of band-limited functions

4.1 The Paley-Wiener class and spline interpolation

If $0 < \beta < \infty$ the Paley-Wiener class PW_β consists of those functions in $L^2(-\infty, \infty)$ whose Fourier transforms have support in the interval $[-\beta, \beta]$. In other words

$$PW_\beta = \left\{ f \in L^2(-\infty, \infty) : \ \operatorname{supp} \hat{f} \subset [-\beta, \beta] \right\}.$$

We remind the reader that such functions f are restrictions to the real line of entire functions $F(z)$ of exponential type β. In particular such f's are continuous and well defined pointwise.

Suppose $\{x_n\}$ is a sequence which satisfies condition RRB. If f in PW_π then the sequence of values $\{f(x_n)\}$ is in $\ell^2(\mathbb{Z})$. Define $S_k f$ to be the spline in $SH_k(\{x_n\})$ which interpolates f on $\{x_n\}$, namely,

$$S_k f(x_n) = f(x_n) \quad \text{for all} \quad n \in \mathbb{Z}.$$

In view of Proposition 2 the function $S_k f$ is well defined; that is, $S_k f$ exists and is unique.

The remainder of this paper is devoted to showing that

$$\lim_{k \to \infty} S_k f(x) = f(x)$$

uniformly and in $L^2(-\infty, \infty)$. Thus in what follows we always assume that $\{x_n\}$ is an RRB sequence, f is in PW_π, and $S_k f$ is the element in $SH_k(\{x_n\})$ which interpolates f on $\{x_n\}$.

4.2 Elementary properties of $S_k f$

In view of the exponential decay of the λ_n's and the fact that $\mathcal{E}(\{x_n\})$ is a Riesz basis for $L^2([-\pi, \pi])$ it should be clear that the mapping $f \to S_k f$ is a linear transformation from PW_π to $L^2(-\infty, \infty)$ which satisfies

$$\|S_k f\|_{L^2(-\infty,\infty)} \leq c \|\{f(x_n)\}\|_{\ell^2(\mathbb{Z})} \leq C \|f\|_{L^2(-\infty,\infty)} \tag{22}$$

where c and C are positive constants independent of f.

Since $D^{2k} S_k f(x) = \sum_{n \in \mathbb{Z}} a_n \delta(x - x_n)$ we have

$$\xi^{2k} \widehat{S_k f}(\xi) = \Psi_k(\xi) \tag{23}$$

where

$$\Psi_k(\xi) = (-1)^k \sum_{n\in\mathbb{Z}} a_n e^{-ix_n\xi}. \tag{24}$$

$\widehat{S_k f}(\xi)$ is in $L^2(-\infty,\infty)$ implies that $\Psi_k(\xi)$ is locally in L^2 and, in particular, is in $L^2([-\pi,\pi])$. Because of (24) and the fact that $\{x_n\}$ is an RRB sequence we may conclude that

$$\|\Psi_k\|_{L^2([(2j-1)\pi,\,(2j+1)\pi])} \le C\|\Psi_k\|_{L^2([-\pi,\pi])}$$

where C is a constant independent of j and that $\sum_{n\in\mathbb{Z}}|a_n|^2 < \infty$. For later reference we denote the restriction of Ψ_k to the interval $[-\pi,\pi]$ by ψ_k; in other words $\psi_k = \Psi_k|_{[-\pi,\pi]}$ and if $(2j-1)\pi \le \xi \le (2j+1)\pi$ then

$$\Psi_k(\xi) = A^j\psi_k(\xi - 2\pi j) \tag{25}$$

for all j in $\mathbb{Z}$ where A^j is the prolongation operator defined in the previous section.

In view of (23) we may write

$$\widehat{S_k f}(\xi) = \frac{\Psi_k(\xi)}{\xi^{2k}} \tag{26}$$

and

$$\|\widehat{S_k f}(\xi)\|^2_{L^2(-\infty,\infty)} = \|\widehat{S_k f}(\xi)\|^2_{L^2([-\pi,\pi])} + \sum_{j\in\mathbb{Z}\setminus\{0\}} \int_{(2j-1)\pi}^{(2j+1)\pi} \left|\frac{\Psi_k(\xi)}{\xi^{2k}}\right|^2 d\xi,$$

which by virtue of (25) may be re-expressed as

$$\|\widehat{S_k f}(\xi)\|^2_{L^2(-\infty,\infty)} = \|\widehat{S_k f}(\xi)\|^2_{L^2([-\pi,\pi])} + \sum_{j\in\mathbb{Z}\setminus\{0\}} \int_{-\pi}^{\pi} \left|\frac{A^j\psi_k(\xi)}{(\xi+2\pi j)^{2k}}\right|^2 d\xi \tag{27}$$

Proposition 3 *The function ψ_k satisfies the relation*

$$\frac{\psi_k(\xi)}{\xi^{2k}} + \sum_{j\in\mathbb{Z}\setminus\{0\}} A^{*j}\left(\frac{1}{(\eta+2\pi j)^{2k}}A^j\psi_k(\eta)\right)(\xi) = \hat{f}(\xi) \tag{28}$$

for $-\pi \le \xi \le \pi$ where A is the prolongation operator and A^* is its adjoint.

Proof: Write

$$\int_{-\pi}^{\pi} \hat{f}(\xi)e^{ix_n\xi}d\xi \;=\; 2\pi f(x_n) = 2\pi S_k f(x_n)$$

$$
\begin{aligned}
&= \int_{-\infty}^{\infty} \frac{\Psi_k(\xi)}{\xi^{2k}} e^{ix_n\xi} d\xi \\
&= \sum_{j\in Z} \int_{(2j-1)\pi}^{(2j+1)\pi} \frac{\Psi_k(\xi)}{\xi^{2k}} e^{ix_n\xi} d\xi \\
&= \sum_{j\in \mathbb{Z}} \int_{-\pi}^{\pi} \frac{\Psi_k(\xi+2\pi j)}{(\xi+2\pi j)^{2k}} e^{ix_n(\xi+2\pi j)} d\xi \\
&= \sum_{j\in \mathbb{Z}} \int_{-\pi}^{\pi} \frac{A^j\psi_k(\xi)}{(\xi+2\pi j)^{2k}} (A^j e^{ix_n\eta})(\xi) d\xi \\
&= \int_{-\pi}^{\pi} \left\{ \sum_{j\in \mathbb{Z}} A^{*j} \Big(\frac{A^j\psi_k(\eta)}{(\eta+2\pi j)^{2k}} \Big)(\xi) \right\} e^{ix_n\xi} d\xi
\end{aligned}
$$

and hence

$$
(29) \quad \int_{-\pi}^{\pi} \hat{f}(\xi) e^{ix_n\xi} d\xi = \int_{-\pi}^{\pi} \left\{ \sum_{j\in \mathbb{Z}} A^{*j} \Big(\frac{1}{(\eta+2\pi j)^{2k}} (A^j\psi_k)(\eta) \Big)(\xi) \right\} e^{ix_n\xi} d\xi.
$$

Since (29) is true for all n in $\mathbb{Z}$ and $\{x_n\}$ is an RRB sequence it follows that

$$
\hat{f}(\xi) = \sum_{j\in \mathbb{Z}} A^{*j} \Big(\frac{1}{(\eta+2\pi j)^{2k}} A^j\psi_k(\eta) \Big)(\xi)
$$

which is the desired result. ∎

For later reference we re-express (28) as

$$
(30) \qquad \widehat{S_k f}(\xi) + \sum_{j\in \mathbb{Z}\setminus\{0\}} A^{*j} \Big(\Big(\frac{\eta}{\pi} + 2j\Big)^{-2k} A^j \frac{\psi_k}{\pi^{2k}}(\eta) \Big)(\xi) = \hat{f}(\xi)
$$

for $-\pi \le \xi \le \pi$. If we use the notation $\hat{s}_k$ to denote the restriction of $\widehat{S_k f}$ to the interval $[-\pi, \pi]$, in other words $\hat{s}_k = \widehat{S_k f}|_{[-\pi,\pi]}$, then (30) may be expressed as

$$
(31) \quad \hat{s}_k(\xi) + \sum_{j\in \mathbb{Z}\setminus\{0\}} A^{*j} \Big(\Big(\frac{\eta}{\pi} - 2j\Big)^{-2k} (A^j \Big[\Big(\frac{\omega}{\pi}\Big)^{2k} \hat{s}_k(\omega) \Big])(\eta) \Big)(\xi) = \hat{f}(\xi)
$$

for $-\pi \le \xi \le \pi$. This expression may be abbreviated to

$$
(32) \qquad (I + B_k M_k)\hat{s}_k = \hat{f}
$$

where $\hat{s}_k$ and $\hat{f}$ are elements of $L^2([-\pi,\pi])$ and I, B_k, M_k are linear operators on $L^2([-\pi,\pi])$ defined by

$$\begin{aligned} B_k g(\xi) &= \sum_{j\in\mathbf{Z}\setminus\{0\}} A^{*j}\Big((\frac{\eta}{\pi}+2j)^{-2k}A^j g(\eta)\Big)(\xi) \\ M_k g(\xi) &= (\xi/\pi)^{2k} g(\xi) \end{aligned}$$

and I is the usual identity operator $Ig(\xi) = g(\xi)$. Note that

$$\|B_k g\|_{L^2([-\pi,\pi])} \leq \Big\{ \sum_{j\in\mathbf{Z}\setminus\{0\}} \frac{C}{(2|j|-1)^{2k}} \Big\} \|g\|_{L^2([-\pi,\pi])}$$

or, more simply,

(33)
$$\|B_k g\|_{L^2([-\pi,\pi])} \leq C\|g\|_{L^2([-\pi,\pi])}$$

where C is a constant independent of k and g.

Lemma 1 *If f is in PW_π and ψ_k is related to $\hat{f}$ via (28) then*

(34)
$$\|\psi_k\|_{L^2([-\pi,\pi])} \leq \pi^{2k}\|\hat{f}\|_{L^2([-\pi,\pi])}$$

Proof: Take the $L^2([-\pi,\pi])$ scalar product of the function ψ_k with both sides of (28). This results in

(35)
$$\langle \frac{\psi_k}{\xi^{2k}},\ \psi_k\rangle + \sum_{j\in\mathbf{Z}\setminus\{0\}} \langle \frac{A^j\psi_k}{(\xi+2\pi j)^{2k}},\ A^j\psi_k\rangle = \langle \hat{f},\ \psi_k\rangle.$$

Since the summands on the left hand side of (35) are non-negative we have

(36)
$$\langle \frac{\psi_k}{\xi^{2k}},\ \psi_k\rangle \leq |\langle \hat{f},\ \psi_k\rangle|.$$

Inequality (36) together with

$$\frac{1}{\pi^{2k}}\|\psi_k\|^2_{L^2([-\pi,\pi])} \leq \langle \frac{\psi_k}{\xi^{2k}},\ \psi_k\rangle$$

and

$$|\langle \hat{f},\ \psi_k\rangle| \leq \|\hat{f}\|_{L^2([-\pi,\pi])}\|\psi_k\|_{L^2([-\pi,\pi])}$$

imply that

$$\frac{1}{\pi^{2k}}\|\psi_k\|^2_{L^2([-\pi,\pi])} \leq \|\hat{f}\|_{L^2([-\pi,\pi])}\|\psi_k\|_{L^2([-\pi,\pi])}$$

which, upon simplification, is the desired result. ∎

Lemma 2 *If f is in PW_π then*

$$\|\widehat{S_k f}\|_{L^2([-\pi,\pi])} \leq C\|\hat{f}\|_{L^2([-\pi,\pi])} \tag{37}$$

where C is a positive constant independent of f and k.

Proof: By virtue of (30) we may write

$$\begin{aligned} \|\widehat{S_k f}\|_{L^2([-\pi,\pi])} &\leq \|\hat{f}\|_{L^2([-\pi,\pi])} \\ &+ \sum_{j\in\mathbf{Z}\setminus\{0\}} \|A^{-j}\left((\frac{\xi}{\pi}+2j)^{-2k}\ A^j\frac{\psi_k}{\pi^{2k}}\right)\|_{L^2([-\pi,\pi])}. \end{aligned} \tag{38}$$

Now

$$\|A^{-j}\left((\frac{\xi}{\pi}+2j)^{-2k}A^j\frac{\psi_k}{\pi^{2k}}\right)\|_{L^2([-\pi,\pi])} \leq \frac{C}{(2|j|-1)^{2k}}\cdot\frac{1}{\pi^{2k}}\cdot\|\psi_k\|_{L^2([-\pi,\pi])}$$

where C is a constant independent of j, k, and f. In view of (34) the last inequality together with (38) imply the desired result. ■

Proposition 4 *The mapping $f \to S_k f$ from PW_π to $L^2([-\infty,\infty])$ is bounded uniformly with respect to k. In other words, if f is in PW_π then*

$$\|S_k f\|_{L^2(-\infty,\infty)} \leq C\|f\|_{L^2(-\infty,\infty)} \tag{39}$$

where C is a positive constant independent of k and f.

Proof: In view of Plancherel's formula (39) is equivalent to

$$\|\widehat{S_k f}\|_{L^2(-\infty,\infty)} \leq C\|\hat{f}\|_{L^2(-\infty,\infty)}. \tag{40}$$

To see (40) use (27), the identity

$$\frac{A^j\psi_k(\xi)}{(\xi+2\pi j)^{2k}} = \left(\frac{\xi}{\pi}+2j\right)^{-2k}(A^j\frac{\psi_k}{\pi^{2k}})(\xi),$$

and the fact that

$$\|\left(\frac{\xi}{\pi}+2j\right)^{-2k}(A^j\frac{\psi_k}{\pi^{2k}})(\xi)\|_{L^2([-\pi,\pi])} \leq \frac{C}{(2|j|-1)^{2k}}\cdot\frac{1}{\pi^{2k}}\|\psi_k\|_{L^2([-\pi,\pi])}$$

to write

$$\begin{aligned} \|\widehat{S_k f}\|^2_{L^2(-\infty,\infty)} &\leq \|\widehat{S^k f}\|^2_{L^2([-\pi,\pi])} \\ &+ \frac{C^2}{\pi^{4k}}\left\{\sum_{j\in\mathbf{Z}\setminus\{0\}}\frac{1}{(2|j|-1)^{4k}}\right\}\|\psi_k\|^2_{L^2([-\pi,\pi])}. \end{aligned} \tag{41}$$

Since the constant C in (41) is independent of f and k, inequalities (34) and (37) together with (41) imply the desired result. ■

4.3 Behavior of $S_k f$ as $k \to \infty$.

Theorem 4 *If f is PW_π then*

$$\lim_{k\to\infty} \|f - S_k f\|_{L^2(-\infty,\infty)} = 0 \tag{42}$$

Proof: By virtue of Plancherel's formula (42) is equivalent to

$$\lim_{k\to\infty} \|\hat{f} - \widehat{S_k f}\|_{L^2(-\infty,\infty)} = 0. \tag{43}$$

To see (43) use the fact that $\hat{f}$ vanishes outside the interval $[-\pi, \pi]$ and write

$$\begin{aligned} \|\hat{f} - \widehat{S_k f}\|^2_{L^2(-\infty,\infty)} &= \|\hat{f} - \widehat{S_k f}\|^2_{L^2([-\pi,\pi])} \\ &+ \sum_{j\in\mathbf{Z}\setminus\{0\}} \|\widehat{S_k f}\|^2_{L^2([(2j-1)\pi,\ (2j+1)\pi])}. \end{aligned} \tag{44}$$

Estimate the size of the first term on the right hand side of (44) as follows: Let $\hat{s}_k$ denote the restriction of $\widehat{S_k f}$ to the interval $[-\pi, \pi]$ and recall relation (32), which is

$$(I + B_k M_k)\hat{s}_k = \hat{f}.$$

In view of Proposition 4 it follows that the operators $I + B_k M_k$ are invertible as mappings from $L^2([-\pi, \pi])$ to itself and the inverses are uniformly bounded, namely

$$\|(I + B_k M_k)^{-1}\|_{L^2([-\pi,\pi])\to L^2([-\pi,\pi])} \leq C \tag{45}$$

where C is a constant independent of k. Thus we may write

$$\hat{f} - \hat{s}_k = \hat{f} - (I + B_k M_k)^{-1}\hat{f} = (I + B_k M_k)^{-1} B_k M_k \hat{f}$$

or more directly

$$\hat{f} - \hat{s}_k = (I + B_k M_k)^{-1} B_k M_k \hat{f}. \tag{46}$$

Recall that the operator norm of B_k is bounded independent of k, see (33). This together with (45) and (46) imply that

$$\|\hat{f} - \widehat{S_k f}\|_{L^2([-\pi,\pi])} \leq C\|M_k \hat{f}\|_{L^2([-\pi,\pi])} \tag{47}$$

which is the desired estimate for the first term.

Estimate the size of the remaining terms on the right hand side of (44) as follows: Use reasoning similar to that used to obtain (27) to write

$$\sum_{j\in\mathbf{Z}\setminus\{0\}} \|\widehat{S_k f}\|^2_{L^2([(2j-1)\pi,(2j+1)\pi])} = \sum_{j\in\mathbf{Z}\setminus\{0\}} \int_{-\pi}^{\pi} \left| \frac{A^j\psi_k(\xi)}{(\xi+2\pi j)^{2k}} \right|^2 d\xi. \tag{48}$$

Note that

$$A^j\psi_k = \pi^{2k} A^j M_k \hat{s}_k \tag{49}$$

where, as above, $\hat{s}_k$ is the restriction of $\widehat{S_k f}$ to the interval $[-\pi,\pi]$ and $M_k s_k(\xi) = (\xi/\pi)^{2k}\hat{s}_k(\xi)$. Thus

$$\frac{A^j\psi_k(\xi)}{(\xi+2\pi j)^{2k}} = \left(\frac{\xi}{\pi}+2j\right)^{-2k} A^j M_k \hat{s}_k(\xi) \tag{50}$$

and the j^{th} term on the right hand side of (48) is equal to

$$\int_{-\pi}^{\pi} \left| \left(\frac{\xi}{\pi}+2j\right)^{2k} A^j M_k \hat{s}_k(\xi) \right|^2 d\xi$$

which is dominated by

$$\left\{ \frac{C}{(2|j|-1)^{2k}} \|M_k \hat{s}_k\|_{L^2([-\pi,\pi])} \right\}^2.$$

Hence we may write

$$\sum_{j\in\mathbf{Z}\setminus\{0\}} \|\widehat{S_k f}\|^2_{L^2([(2j-1)\pi,\,(2j+1)\pi])} \le C\|M_k\hat{s}_k\|^2_{L^2([-\pi,\pi])} \tag{51}$$

where C is independent of f and k. Now

$$\|M_k\hat{s}_k\|_{L^2([-\pi,\pi])} \le \|M_k\hat{f}\|_{L^2([-\pi,\pi])} + \|M_k(s_k-\hat{f})\|_{L^2([-\pi,\pi])} \tag{52}$$

and clearly

$$\|M_k(s_k-\hat{f})\|_{L^2([-\pi,\pi])} \le \|s_k - f\|_{L^2([-\pi,\pi])}.$$

In view of (47) the last inequality may be replaced by

$$\|M_k(s_k-\hat{f})\|_{L^2([-\pi,\pi])} \le C\|M_k\hat{f}\|_{L^2([-\pi,\pi])} \tag{53}$$

where C is a constant independent of f and k. Combining (51), (52), and (53) results in the estimates

$$\|M_k\hat{s}_k\|_{L^2([-\pi,\pi])} \le C\|M_k\hat{f}\|_{L^2([-\pi,\pi])} \tag{54}$$

and

(55) $$\sum_{j\in\mathbf{Z}\setminus\{0\}} \|\widehat{S_k f}\|^2_{L^2([(2j+1)\pi,\,(2j+1)\pi])} \le C\|M_k \hat{f}\|^2_{L^2([-\pi,\pi])}$$

where C is a constant independent of f and k.

Estimates (47) and (55) when substituted into (44) lead to

(56) $$\|\hat{f} - \widehat{S_k f}\|_{L^2(-\infty,\infty)} \le C\|M_k \hat{f}\|_{L^2([-\pi,\pi])}$$

where C is a constant independent of f and k. Inequality (56) implies the desired result since $M_k\hat{f}(\xi) = (\xi/\pi)^{2k}\hat{f}(\xi)$ and the Lebesgue dominated convergence theorem imply that

$$\lim_{k\to\infty} \|M_k f\|_{L^2([-\pi,\pi])} = 0.$$

■

Inequality (56) is interesting and worth re-statement as a corollary of the above argument.

Lemma 3 *If f is in PW_π then*

(57) $$\|f - S_k f\|^2_{L^2(-\infty,\infty)} \le C\int_{-\pi}^{\pi} \left|\left(\frac{\xi}{\pi}\right)^{2k} \hat{f}(\xi)\right|^2 d\xi$$

where C is a constant independent of k and f.

The technique used in the proof of the theorem leads to other convergence results. For example, to estimate the pointwise difference between f and $S_k f$ we may write

(58) $$S_k f(x) - f(x) = \frac{1}{2\pi}\left\{\int_{-\pi}^{\pi}(\widehat{S_k f}(\xi) - \hat{f}(\xi))e^{ix\xi}d\xi + \int_{|\xi|>\pi} \widehat{S_k f}(\xi)e^{ix\xi}d\xi\right\}$$

The first integral on the right hand side of (58) can be estimated via the Schwartz inequality and (47) to get

(59) $$\left|\int_{-\pi}^{\pi}(\widehat{S_k f}(\xi) - \hat{f}(\xi))e^{ix\xi}d\xi\right| \le C\|M_k \hat{f}\|_{L^2([-\pi,\pi])}$$

where C is a constant independent of f and k. To estimate the second integral on the right hand side of (58) use (26) and reasoning similar to that used to get (27) and write

(60) $$\int_{|\xi|>\pi} \widehat{S_k f}(\xi)e^{ix\xi}d\xi = \sum_{j\in\mathbf{Z}\setminus\{0\}} \int_{-\pi}^{\pi} \frac{A^j\psi_k(\xi)}{(\xi + 2\pi j)^{2k}} e^{ix(\xi+2\pi j)}d\xi.$$

Now use (49) and (50) to see that if I_j is the j-th term on the right hand side of (60) it can be expressed as

$$I_j = \int_{-\pi}^{\pi} \left(\frac{\xi}{\pi} - 2j\right)^{-2k} A^j M_k \hat{s}_k(\xi) e^{ix(\xi+2\pi j)} d\xi.$$

Using the Schwartz inequality results in

$$(61) \qquad |I_j| \leq \|A^j M_k \hat{s}_k(\xi)\|_{L^2([-\pi,\pi])} \cdot \left\{ \int_{-\pi}^{\pi} \left|\frac{\xi}{\pi} - 2j\right|^{-4k} d\xi \right\}^{\frac{1}{2}}.$$

Since the first and second terms in the product on the right hand side of (61) are dominated by

$$C\|M_k f\|_{L^2([-\pi,\pi])} \text{ and } \frac{\sqrt{2\pi}}{(2|j|-1)^{2k}}$$

respectively (the first follows from (54) and the second by direct estimation) we have

$$(62) \qquad |I_j| \leq \frac{C}{(2|j|-1)^{2k}} \|M_k \hat{f}\|_{L^2([-\pi,\pi])}$$

where C is independent of j, k, and f. Thus

$$\begin{aligned} \left|\int_{|\xi|>\pi} \widehat{S_k f}(\xi) e^{ix\xi} d\xi\right| &\leq \sum_{j\in\mathbb{Z}\setminus\{0\}} |I_j| \\ &\leq \left\{ \sum_{j\in\mathbb{Z}\setminus\{0\}} \frac{C}{(2|j|-1)^{2k}} \right\} \|M_k \hat{f}\|_{L^2([-\pi,\pi])} \end{aligned}$$

which may be simplified to

$$(63) \qquad \left|\int_{|\xi|>\pi} \widehat{S_k f}(\xi) e^{ix\xi} d\xi\right| \leq C\|M_k \hat{f}\|_{L^2([-\pi,\pi])}$$

where C is a constant independent of k and f.

Combining (59) and (63) to estimate the integrals on the right hand side of (58) leads to a bound on $|S_k f(x) - f(x)|$. We summarize this as follows.

Theorem 5 *If f is in PW_π then for all x in $\mathbb{R}$*

$$(64) \qquad |S_k f(x) - f(x)| \leq C \left\{ \int_{-\pi}^{\pi} \left| \left(\frac{\xi}{\pi}\right)^{2k} \hat{f}(\xi) \right|^2 d\xi \right\}^{\frac{1}{2}}$$

where C is a constant independent of k and f. It follows that

$$(65) \qquad \lim_{k\to\infty} S_k f(x) = f(x)$$

uniformly on $\mathbb{R}$.

Estimates (57) and (64) lead to even better convergence results if the support of $\hat{f}$ is properly contained in the interval $(-\pi, \pi)$. The rate of convergence is geometric and depends on the size of the support. In particular if the support of $\hat{f}$ is contained in the interval $[-\beta, \beta]$ where $0 < \beta < \pi$ then the integral on the right hand side of (57) and (64) is dominated by

$$\left(\frac{\beta}{\pi}\right)^{2k} \|\hat{f}\|_{L^2(-\infty,\infty)} = 2\pi \left(\frac{\beta}{\pi}\right)^{2k} \|f\|_{L^2(-\infty,\infty)}.$$

We summarize this remark as follows:

Corollary 1 *If f is in PW_β for some β which satisfies $0 < \beta < \pi$ then*

$$\|f - S_k f\|_{L^p(-\infty,\infty)} \leq C \left(\frac{\beta}{\pi}\right)^{2k} \|f\|_{L^2(-\infty,\infty)} \tag{66}$$

for all p which satisfy $2 \leq p \leq \infty$ where C is a constant independent of k and f.

5 Remarks and acknowledgements

In the next three paragraphs we collect several remarks and indicate further references which may be helpful. No attempt is made to be exhaustive.

The theory of splines and their applications has undergone extensive development in the past thirty or so years. In addition to the references cited above we mention that [25] is a relatively recent treatise on the subject which contains an extensive list of references. We note that our version of Proposition 2 is an easy consequence of the material in [3] and bring attention to the fact that the statement of a version with a less restrictive condition on the sequence of knots $\{x_n\}$ can be found in [11].

A history and contemporary exposition of the theory of Riesz bases consisting of exponentials, including the results quoted in Section 3.1, is given in the survey [10]. See also the textbook [27]. More recent surveys and lists of references may be found in [1, 6]. Material on mean periodic functions may be found in [13, 26].

The fact that various classes of band-limited functions, or entire functions of exponential type, can be recovered from their samples on certain discrete sets is well known. The general subject area is often referred to as sampling theory. For surveys which include the result involving condition (11) mentioned in Subsection 3.2 and a list of further references see [1, 6]; indeed most of the work cited in the previous paragraph is directly related to this subject. Interesting surveys which include various extensions, generalizations, error estimates, numerical methods, and further references can be found in [5, 7, 9].

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Orthogonality Criteria for Compactly Supported Scaling Functions

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Abstract

We give a new simple proof of several criteria for the orthogonality of the integer translates of a scaling function.

A scaling function is a function $\varphi \in L^2(\mathbb{R})$ that satisfies

(1) $$\varphi(x) = \sum_{k \in \mathbb{Z}} c_k \, q \, \varphi(qx - k)$$

for some integer q, $|q| > 1$, and coefficients $c_k \in \mathbb{C}$, and such that the integer translates $\varphi(x-k), k \in \mathbb{Z}$, are an orthonormal system. In order to construct a *compactly supported* scaling function φ [5], one chooses a finite sequence of non-zero coefficients $c_k \in \mathbb{C}$, $c_k = 0$ for $|k| > N$, such that

(2) $$q \sum_{k \in \mathbb{Z}} c_{k-q\ell} \bar{c}_k = \delta_\ell \quad \text{for all} \quad \ell \in \mathbb{Z}$$

and

(3) $$\sum_{k \in \mathbb{Z}} c_k = 1$$

If $m(\xi)$ denotes the corresponding trigonometric polynomial

(4) $$m(\xi) = \sum_{k=-N}^{N} c_k e^{2\pi i k \xi} ,$$

properties (2) and (3) are equivalent to

(5) $$\sum_{j=0}^{q-1} \left| m\left(\xi + \frac{j}{q}\right)\right|^2 = 1 \quad \text{for all} \quad \xi \in \mathbb{R}$$

and

(6) $$m(0) = 1 .$$

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The scaling function φ is then obtained from the infinite product

$$\hat{\varphi}(\xi) = \prod_{j=1}^{\infty} m(q^{-j}\xi) \tag{7}$$

and Fourier transform $\varphi(x) = \int_{\mathbb{R}} \hat{\varphi}(\xi) e^{2\pi i \xi x} d\xi$. The infinite product converges uniformly on compact sets to $\hat{\varphi} \in L^2(\mathbb{R})$, $\varphi(x)$ has compact support, and satisfies the scaling relation (1), see for instance [5], Ch. 6.

The question whether the integer translates of such a φ are mutually orthogonal is more subtle. Conditions (2) and (3) are necessary, but not sufficient to imply orthogonality.

In this article we give a new and simple proof of several orthogonality criteria obtained by A. Cohen, W. Lawton, J. P. Conze and A. Raugi, and Q. Sun.

Since "almost all" sequences satisfying (2) and (3) lead to orthogonal translates, we characterize the case of non-orthogonal translates.

The non-trivial part of the proof will consist in the analysis of the operator

$$Tf(\xi) = \sum_{j=0}^{q-1} \left| m\left(\frac{\xi + j}{q}\right)\right|^2 f\left(\frac{\xi + j}{q}\right).$$

By (5) the constant 1 is always an eigenfunction for the eigenvalue 1.

Theorem 1 *Let $m(\xi)$ be a trigonometric polynomial satisfying (5) and (6) and $\hat{\varphi}(\xi) = \prod_{j=1}^{\infty} m(q^{-1}\xi) \in L^2(\mathbb{R})$. Then the following statements are equivalent:*

(A) The translates $\varphi(x - k)$, $k \in \mathbb{Z}$, are not orthogonal.

(B) There exists a nonconstant nonnegative trigonometric polynomial f so that $Tf = f$.

(C) There exists a $\xi \in (0, 1)$ so that $q^N \xi \equiv \xi (\mathrm{mod}\ 1)$ for some $N > 1$ and

$$|m(q^j\xi)| = 1 \quad \text{for all} \quad j \geq 0.$$

(D) There exist integers $N, s > 0$, $1 \leq s \leq q^N - 1$ so that

$$m\left(\frac{q^j s}{q^N - 1} + \frac{\ell}{q}\right) = 0 \quad \text{for all} \quad j \geq 0 \quad \text{and} \quad \ell = 1, 2, \ldots, q-1$$

(E) There exists $\eta \in (1, 2)$ with the following property: For all $k \in \mathbb{Z}$ there is a $j(k) \geq 1$ such that

$$m\left(q^{-j(k)}\left(\eta + k\right)\right) = 0$$

Remarks 1. The equivalence (A) $\Leftrightarrow$ (E), due to A. Cohen [1], was the first known characterization of orthogonality. It is usually formulated positively in the following way: The integer translates of φ are orthonormal if and only if there exists a compact set $K \subseteq I\!R$ which contains a neighborhood of 0 and satisfies $\cup_{k\in \mathbf{Z}}(k+K) = I\!R$, $(k+K)\cap K = \emptyset$ for $k \neq 0$, such that $m(q^{-j}\xi) \neq 0$ for all $j \geq 1$, and $\xi \in K$ ("m satisfies Cohen's condition").

The characterization (A) $\Leftrightarrow$ (B) was found independently by W. Lawton [8] and J. P. Conze and A. Raugi [4]. The arithmetic characterizations (C) and (D) appear in A. Cohen and Q. Sun [2], and (A) $\Leftrightarrow$ (C) can also be found in [4]. The analysis of eigenfunctions of T also figures prominently in [3].

2. The equivalence (A) $\Leftrightarrow$ (B) $\Leftrightarrow$ (E) extends to more general filters m and also holds in higher dimensions [1, 8, 9].

3. Our contribution is a new organization of the proof which makes all but one implication trivial and a new argument for the crucial step from (B) to (C). Compared with the proofs in the references above or in [5] our proof is simpler and much shorter. ■

Proof: We first remark that by Poisson's summation formula

$$(8)\qquad \sum_{k\in \mathbf{Z}} |\hat{\varphi}(\xi-k)|^2 = \sum_{k\in \mathbf{Z}} \left(\int_{\mathbf{R}} \varphi(x)\overline{\varphi(x-k)}\, dx \right) e^{2\pi i k\xi}\,.$$

If φ has compact support, then $f(\xi) := \sum_{k\in \mathbf{Z}} |\hat{\varphi}(\xi-k)|^2$ is a trigonometric polynomial, and the translates $\varphi(x-k)$ are orthonormal if and only if $f(\xi) = 1$ for all ξ.

(A) $\Rightarrow$ (B). If $\{\varphi(x-k),\ k \in \mathbb{Z}\}$ is not orthonormal, then $f(\xi)$ is a nonconstant nonnegative trigonometric polynomial. Using $\hat{\varphi}(\xi) = m(\frac{\xi}{q})\hat{\varphi}(\frac{\xi}{q})$, we obtain

$$\begin{aligned}(9)\quad f(\xi) &= \sum_{k} |\hat{\varphi}(\xi-k)|^2 = \sum_{j=0}^{q-1} \sum_{\ell\in \mathbf{Z}} \left| m\left(\frac{\xi-\ell q-j}{q}\right)\right|^2 \hat{\varphi}\left(\frac{\xi-\ell q-j}{q}\right) \\ &= \sum_{j=0}^{q-1} \left| m\left(\frac{\xi+j}{q}\right)\right|^2 f\left(\frac{\xi+j}{q}\right) = Tf(\xi)\end{aligned}$$

(B) $\Rightarrow$ (C). Set $m_n(\xi) = \prod_{j=0}^{n-1} m(q^j\xi)$. Then

$$(10)\qquad T^n f(\xi) = \sum_{k=0}^{q^n-1} \left| m_n\left(\frac{\xi+k}{q^n}\right)\right|^2 f\left(\frac{\xi+k}{q^n}\right)$$

as is easily verified by induction. Since $T1 = 1$, we also obtain

$$(11)\qquad T^n 1 = \sum_{k=0}^{q^n-1} \left| m_n\left(\frac{\xi+k}{q^n}\right)\right|^2 = 1 \quad \text{for all } \xi.$$

Now assume that $Tf = f$ for some nonconstant nonnegative trigonometric polynomial. Then one of the extrema is assumed at an interior point $\xi_0, 0 < \xi_0 < 1$. Without loss of generality we may assume $f(\xi) \leq f(\xi_0)$ for all $\xi \in I\!R$. The hypothesis $T^n f = f$ for all n and (11) yield

$$(12) \qquad 0 = f - T^n f = \sum_{k=0}^{q^n - 1} \left| m\Big(\frac{\xi_0 + k}{q^n}\Big)\right|^2 \Big(f(\xi_0) - f(\frac{\xi_0 + k}{q^n})\Big)$$

As all terms in (12) are nonnegative, either $f(\xi_0) = f(\frac{\xi_0+k}{q^n})$ or $m_n(\frac{\xi_0+k}{q^n}) = 0$ holds for each k.

Let $\mathcal{M}$ be the *finite* set of maxima of f and $C =$ card $\mathcal{M}$. Then

$$(13) \qquad \{\eta : \eta = \frac{\xi_0 + k}{q^n} \text{ and } m\Big(\frac{\xi_0 + k}{q^n}\Big) \neq 0\} \subseteq \mathcal{M}$$

Consequently for each n the sum in (11) contains at most C non-zero terms and there is a k_n, $0 \leq k_n < q^n$, such that

$$(14) \qquad \left| m_n\Big(\frac{\xi_0 + k_n}{q^n}\Big)\right| \geq C^{-\frac{1}{2}}$$

We can now choose an infinite subsequence $n(i)$ so that

$$(15) \qquad \frac{\xi_0 + k_{n(i)}}{q^{n(i)}} = \xi \in \mathcal{M}, \xi \neq 0, 1 \quad \text{for all } \; i\,.$$

Since $q^{n(i)}\xi = \xi_0 + k_{n(i)} \equiv \xi_0(\bmod 1)$, there exists an $N > 1$ such that $q^N\xi \equiv \xi(\bmod 1)$, e.g. $N = n(2) - n(1)$.

Finally, since $q^{j+lN}\xi \equiv q^j\xi(\bmod 1)$ for all $l, j \geq 0$ and since the sequence $|m_n(\xi)|, n = 1, 2, \ldots$, is decreasing, we obtain for $L \geq 1$

$$(16) \qquad \prod_{j=0}^{N-1} |m(q^j\xi)| = \Big(\prod_{j=0}^{LN-1} |m(q^j\xi)|\Big)^{1/L} \geq$$

$$\geq \varliminf_{i\to\infty} \left|\prod_{j=0}^{n(i)-1} |m(q^j\xi)\right|^{1/L} = \varliminf_{i\to\infty}\left| m_{n(i)}(\xi)\right|^{1/L} \geq C^{-\frac{1}{2L}}$$

As L was arbitrary, we obtain the desired conclusion $|m(q^j\xi)| = 1$ for all $j \geq 0$.

(C) $\Rightarrow$ (D). As $q^N\xi \equiv \xi$ (mod 1), we have $q^n\xi = \xi + s$ for some s, $1 \leq s \leq q^N - 1$ and thus $\xi = \frac{s}{q^N - 1}$. By (5) and by the assumption $|m(q^j\xi)| = 1$ we obtain $m(\frac{q^j s}{q^N - 1} + \frac{\ell}{q}) = 0$ for $\ell = 1, \ldots q - 1$.

(D) $\Rightarrow$ (E). Set $\eta = \frac{s}{q^N - 1} + 1$. We will verify that for each $k \in \mathbb{Z}$ there is a $J \in \mathbb{Z}$, $J \geq 0$, such that

$$(17) \qquad \frac{\eta + k}{q^{J+1}} = \frac{q^r s}{q^N - 1} + \frac{\ell}{q} + M$$

for some $r \in \{0,1,\ldots,N-1\}$, $\ell \in \{1,\ldots,q-1\}$ and $M \in \mathbb{Z}$. If (17) holds, then by hypothesis $m(\frac{\eta+k}{q^{J+1}}) = 0$.

Given $k \in \mathbb{Z}$, we consider the sequence

$$a_j = q^{-j}\Big(-s \sum_{a=0}^{[\frac{j}{N}]} q^{aN} + k + 1\Big) = -s\frac{q^{N[\frac{j}{N}]+N} - 1}{q^j(q^N-1)} + \frac{k+1}{q^j}, j = 0, 1, \ldots$$

where $[x]$ denotes the smallest integer $\leq x$. The accumulation points of $\{a_j\}$ are the numbers $-\frac{q^r s}{q^N-1} \notin \mathbb{Z}, r = 0, 1, \ldots, N-1$. Therefore there is a largest integer $J \geq 0$, such that $a_J \in \mathbb{Z}$, but $a_{J+1} \notin \mathbb{Z}$, i.e., $a_{J+1} = \frac{l}{q} + M$ for some $l \in \{1,2,\ldots,q-1\}$ and $M \in \mathbb{Z}$. Rewriting a_{J+1} as

$$a_{J+1} = \frac{1}{q^{J+1}}\Big(-s\sum_{a=0}^{[\frac{J}{N}]} q^{aN} + k + 1\Big) = \frac{1}{q^{J+1}}\Big(\frac{s}{q^N-1} + k + 1\Big) - \frac{q^{N[\frac{J}{N}]+N-J-1}s}{q^N-1} = \frac{\ell}{q} + M$$

yields (17).

(E) $\Rightarrow$ (A). If for all $k \in \mathbb{Z}$ $m(q^{-j}(\eta+k)) = 0$ for some $j \geq 1$, then $\hat{\varphi}(\eta+k) = \prod_{j=1}^{\infty} m(q^{-j}(\eta+k)) = 0$ for all k. Thus

$$\sum_{k\in\mathbb{Z}} |\hat{\varphi}(\eta+k)|^2 = 0$$

and the translates $\varphi(x-n)$ cannot be orthonormal by the remark at the beginning of the proof. ∎

The theorem furnishes a quick proof for the classification of self-similar tiles and Haar bases of $L^2(\mathbb{R})$ which was obtained in [6]. Given $q \in \mathbb{Z}, |q| > 1$, let $\mathcal{D} = \{k_0, k_1, \ldots, k_{q-1}\}$ be a complete residue system modulo q, i.e. $k_i \equiv i \pmod q$, and set $c_k = 1/q$ for $k \in \mathcal{D}$ and $c_k = 0$ otherwise. Set

$$Q = Q(\mathcal{D}) = \{x \in \mathbb{R} : x = \sum_{j=1}^{\infty} q^{-j}\epsilon_j, \epsilon_j \in \mathcal{D}\}\,, \tag{18}$$

then Q is a compact and self-similar set satisfying

$$qQ = \bigcup_{l=0}^{q-1}(k_l + Q)\,.$$

Therefore its characteristic function $\phi = \chi_Q$ is the (unique) solution of $\phi(x) = \sum_{l=0}^{q-1}\phi(qx - k_l)$. $\{\phi(x-k), k \in \mathbb{Z}\}$ is orthonormal, if and only if $Q \cap (k+Q)$ has measure 0 for all $k \neq 0$, in other words, if and only if Q tiles $\mathbb{R}$.

In the following gcd denotes the greatest common divisor of a given set of integers.

Theorem 2 ([6]) *The integer translates $k+Q, k\in\mathbb{Z}$, are mutually disjoint (up to sets of measure zero) if and only if gcd $_{i,j}(k_i-k_j)=1$.*

Proof: By the orthogonality criterion $\int \chi_Q(x)\chi_Q(x-k)\,dx \neq 0$ for some $k\neq 0$, if and only if there exists $\xi\in(0,1)$, such that $|m(q^j\xi)|=1$ for all $j\geq 0$. In this case

$$|m(\xi)| = |\frac{1}{q}\sum_{l=0}^{q-1} e^{2\pi i k_l \xi}| = 1$$

implies that $e^{2\pi i k_l\xi}=e^{2\pi i\alpha}$ for some $\alpha\in[0,1]$ and $l=0,1,\ldots,q-1$. Since $k_l\xi\equiv\alpha \pmod 1$, it follows that $(k_i-k_j)\xi\in\mathbb{Z}$ for all $i,j=0,1,\ldots,q-1$ and by taking appropriate linear combinations also gcd $_{i,j}(k_i-k_j)\xi\in\mathbb{Z}$. Since $0<\xi<1$, we obtain gcd $_{i,j}(k_i-k_j)>1$.

Conversely, if $d=$ gcd $_{i,j}(k_i-k_j)>1$, then $|m(q^j\frac{1}{d})|^2=1$ for all $j\geq 0$. ∎

It is now easy to obtain explicit compactly supported wavelet bases that are analogous to the ordinary Haar bases. See [7] for more details.

Corollary 1 ([7]) *Let q, k_l, Q be as above and assume that gcd $_{i,j}(k_i-k_j)=1$. Let $U=(u_{i,j})_{i,j=0,\ldots,q-1}$ be a unitary $q\times q$-matrix, such that $u_{0j}=q^{-1/2}$, $j=0,\ldots,q-1$. Define*

$$\psi_i(x)=\sum_{j=0}^{q-1} u_{ij}q^{1/2}\chi_Q(qx-k_j) \quad \textit{for } i=1,\ldots,q-1 \ . \tag{19}$$

Then the collection of functions

$$q^{j/2}\psi_i(q^jx-k) \qquad j,k\in\mathbb{Z},\ i=1,\ldots,q-1$$

is a complete orthonormal basis for $L^2(\mathbb{R})$.

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